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Algebra II

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Problem Set # 12

Exercise 1:
Let $\mathcal{X} = \{e_1, e_2\}$ be the standard basis in the complex inner product space $V = \mathbb{C}^2$ equipped with the usual Euclidean inner product $(a, b) = a_1\bar{b_1} + a_2\bar{b_2}$. Let $\mathcal{N} = \{f_1, f_2\}$ be the basis such that $f_1 = e_1$ and $f_2 = e_1 + e_2$ and define $T : V \to V$ to be the linear operator such that $T(f_1) = f_1$ and $T(f_2) = \frac{1}{2}f_2$. Obviously $\mathcal{N}$ diagonalizes T but the basis eigenvectors f_1, f_2 are not orthogonal.

1. Find $[T]_{\mathcal{X},\mathcal{X}}$ and $[e^{tT}]_{\mathcal{X},\mathcal{X}}$ for $t \in \mathbb{R}$.
2. Explain why T cannot be a self-adjoint operator ($T^* = T$).
3. Explain why there cannot be an ON basis in V that diagonalizes T.
4. Find the solution $X(t)$, $t \in \mathbb{R}$, of the vector-valued differential equation
$$\frac{dX}{dt} = A \cdot X(t) \; with \; X(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
where $A = [T]_{\mathcal{X},\mathcal{X}}$.

Exercise 2:
In $V = \mathbb{R}^3$ equipped with the usual Euclidean inner product let $M = (\mathbb{R}f_3)^\perp$ where $f_3 = (1, -2, 3)$.

1. Give a formula $(y_1, y_2, y_3) = R(x_1, x_2, x_3)$ in terms of inner products for the linear operator $R : \mathbb{R}^3 \to \mathbb{R}^3$ that reflects vectors across the plane M. Find the image of the particular vector $R(1, 1, -3)$.
2. Prove that R is an isometry, so that
$$||R(x) - R(y)|| = ||x - y||$$
and hence is a bijection $R : \mathbb{R}^3 \to \mathbb{R}^3$.
3. Find an ON basis $\{e_1, e_2\}$ for M.
4. What is the matrix $[R]_{\mathcal{N},\mathcal{N}}$ with respect to the ON basis $\mathcal{N} = \{e_1, e_2, e_3\}$ such that $e_3 = f_3/||f_3||$?
5. Is $R : V \to V$ orthogonally diagonalizable? Explain.

Exercise 3:
Let $V = \mathbb{C}^2$ with the usual Euclidean inner product and let $T : V \to V$ be the linear operator whose matrix with respect to the standard ON basis $\mathcal{X} = \{e_1, e_2\}$ is

$$[T]_{\mathcal{X},\mathcal{X}} = \begin{pmatrix} 7 & \sqrt{3} \\ \sqrt{3} & 5 \end{pmatrix}$$

1. Determine the spectrum $sp_{\mathbb{C}}(T) = \{\lambda_1, \lambda_2\}$ and show T is orthogonally diagonalizable by finding an ON basis $\mathcal{N} = \{f_1, f_2\}$ of eigenvectors.

2. Explain why T is self-adjoint.

If an operator is self-adjoint (or merely diagonalizable) it has a spectral decomposition

$$T = \sum_{\lambda \in sp(T)} \lambda P_\lambda$$

where $P_\lambda =$ (projection onto E_λ along $\oplus_{\nu \neq \lambda} E_\nu$) which describes T as a weighted sum of projections onto the eigenspaces.

3. Find the matrices $[P_{\lambda_k}]$ that describe the spectral projections with respect ti the diagonalizing basis $\mathcal{N}$.
   Hint: You only need to find one of these matrices because $P_{\lambda_1} + P_{\lambda_2} = I \Rightarrow P_{\lambda_2} = I - P_{\lambda_1}$.

4. Find the matrices $[P_{\lambda_k}]$ that describes the spectral projections with respect to the standard ON basis $\mathcal{X}$.
   Hint: Again, you only need to find one of these matrices.

5. In terms of the spectral decomposition the square root of T is the operator given by
   $$\sqrt{T} = \sum_{\lambda \in sp(T)} \sqrt{\lambda} P_\lambda$$
   Find the matrix $[\sqrt{T}]$ that describes $\sqrt{T}$ with respect to the standard ON basis.